2018 SINGAPORE-CAMBRIDGE A LEVEL H2 MATH P1 SUGGESTED ANSWER KEY (9758)

Written and Prepared by Mr Mitch Peh

Preface

Dear JC students in Singapore,
Hope you will find this A Level examination solution set useful for your revision.

The answers and comments to this solution set are personally crafted and written by Mr Mitch Peh, an experienced former MOE JC lecturer and tutor in Singapore. Currently, Mr Peh is a full time A Level private tutor, specialising in the teaching of A Level subjects: Physics, Chemistry, Mathematics and Economics at both H1 and H2 Levels. You can find the A Level solutions for the other subjects under the various subject tabs at www.jcpcme.com.

Mr Peh has a proven track record in helping his students achieve success for the A Levels and internal school examinations including promos, advancement tests to JC2, block tests, mid years and prelims. Most of Mr Peh's students achieve "A's and 'B's grades for the A Level examinations. During his stint teaching at St Andrew's Junior College, Mr Peh has helped his classes achieve 100% promotion to JC2 on multiple occasions, attain close to 100% "A"s for H1 Project Work, clinch accolades like "Most Improved Class Award" and "Best Performing Class of the Cohort" for many of the internal school examinations. Mr Peh also has former students who subsequently went on to pursue H3 subjects and enroll in prestigious university courses like Dentistry, Medicine and Law.

If you are interested to be coached by Mr Peh for your preparations towards the A Levels, these are 3 more reasons why you should join Mr Peh's classes:

1. **Lessons can be fully customised to your needs**
- You have the full autonomy to decide the subject(s), content and pace that you want to cover for each lesson, out of any of the 4 subjects: Physics, Chemistry, Mathematics or Economics.
- Mr Peh will help to analyse your weaknesses in each individual subject and provide personalised feedback and suggestions for improvement.

2. **Answers to your questions can be addressed outside of the classroom**
- If you face any difficulty or challenge doing any of your tutorial questions, simply take a screenshot with your phone and send it to Mr Peh via Whatsapp. Mr Peh will answer your questions in the earliest possible time when he is available.

3. **You only pay the price of 1 subject but enjoy premium coverage for all 4 subjects**.
- Mr Peh provides resources for all 4 subjects including summarised notes, compiled topical questions sourced from past year school prelim examinations, Practical guides for Chemistry & Physics, examination checklists, mock papers etc.
- This is probably the only tuition service in Singapore which allows you to enjoy such extensive coverage and benefits.

Note that Mr Peh only takes in a limited number of students each year. You can find his lesson slots available under "Tuition Services" tab at www.jcpcme.com. For any further enquiries, you can directly whatsapp him at 9651 7737.

For the solution set below, if you find any discrepancies or you have any feedback or comments, please kindly direct them to Mr Peh through Whatsapp at 9651 7737.

The question paper has been omitted due to copyright reasons.

Overall Remarks

- This is a difficult paper where mathematical manipulations can be complicated at times, strong problem solving skills is required and you need to be careful to avoid careless mistakes.
- To do well for this paper you need to be particularly strong with Calculus as there is quite a heavy emphasis on it in this paper.
  - Differentiation: Q1, Q3, Q7, Q8, Q9, Q10
  - Integration: Q1, Q2, Q3, Q8, Q9
- Strong problem solving skills is required because the method and approach may not be obvious for some parts of the questions,
  - For Q1(ii), you will need to think about how to make use of the results from (i) which may not be obvious so some students may not be able to see it.
  - For Q5, you need to realise that you should perform long division for the function of f so that you can deduce the value of b.
  - For both parts of Q6, it may not be obvious to some students on how to arrive at the desired results involving vectors.
  - For Q9(i), students may face difficulties with the manipulation to show the expression $\frac{dy}{dx} = \cot\theta$
- In this paper, getting the correct results for the earlier part of the questions are also important. Making careless mistakes will likely cause us to lose significant amount of marks in the later part of the questions since we will not have the correct expressions for manipulation. This applies especially to questions 1 to 4, question 6 and 11.
- In terms of O level knowledge, being familiar with double angle formulae is important to answer question 9, being familiar to find the equations of tangents and normal to curve are important to answer question 7 and 9.
- Some of A Level commonly tested concepts are repeated in this paper, including self-inverse function in question 5, dealing with parametric equations in question 9.
- Similar to the 2017 A Level Paper 1, the last 2 questions are contextual based questions. The topic of differential equations has been used as a contextual question again while the other topic here is APGP related to finance.
- Moving forward to future A level H2 Math Paper 1, you can continue to expect the last 2 questions to be contextual questions likely to be based on topics such as differential equations, vectors, APGP, maximisation and minimisation problems.
- I would consider question 10 to be the most difficult question in this paper as the mathematical manipulations involving differentiation is challenging with the sheer number of arbitrary values provided.
- For question 11, it is important to form the correct arithmetic series and geometric series expressions based on the information provided in the question. Make sure you double check before continuing with the manipulation.
- We should also pay attention to the topics that have not been tested and topics which have been tested to only a small extent so that we can better prepare ourselves for Paper 2.
  - Topics not tested: Complex Numbers, Maclaurin Series, Graph Transformation
  - Topics tested to only a small extent: Vectors (3 dimensional vector geometry has not been tested)

2018 A Level 9758 H2 Math P1 Suggested Solution

1.	(i)	Topic: Differentiation on the application of product rule Given $y=\frac{\ln x}{x}$, Suppose we differentiate ln*x* first, followed by differentiating $\frac{1}{x}$. $\frac{dy}{dx}=\frac{1}{x^2}-\frac{\ln x}{x^2}$ [A2] <u>Comments</u> • An alternative approach is to apply quotient rule to obtain the expression for $\frac{dy}{dx}$, where we should remember to differentiate the expression in the numerator first.	[2]
	(ii)	Topic: Integration using differentiation results from above $\int_1^e \frac{\ln x}{x^2}\,dx=-\int_1^e \frac{1}{x^2}-\frac{\ln x}{x^2}-\frac{1}{x^2}\,dx$ [M1] $=-\int_1^e \frac{1}{x^2}-\frac{\ln x}{x^2}\,dx+\int_1^e \frac{1}{x^2}\,dx$ Applying the result from (i), we have: $=-\left[\frac{\ln x}{x}\right]_1^e-\left[\frac{1}{x}\right]_1^e$ [M1] $=-\left(\frac{1}{e}-0\right)-\left(\frac{1}{e}-1\right)=1-\frac{2}{e}$ [M2] <u>Comments</u> • There is no need to apply integration by parts, just introduce an additional term of $\frac{1}{x^2}$ to use the result from (i). • Be careful when working with the signs here. • We can check our answer with the use of the GC since this question involves definite integrals.	[4]

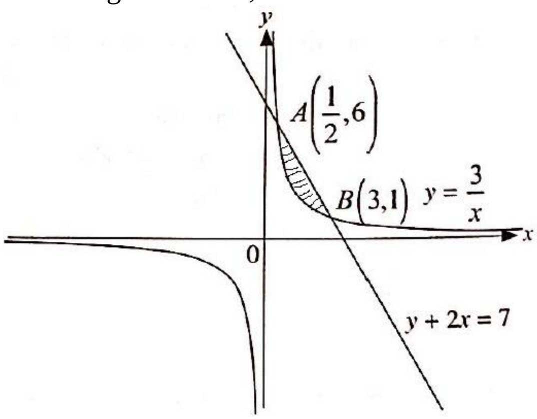
2	(i)	Find the *x*-coordinates of A and B. Topic: Solving equations Given $y=\frac{3}{x}$ …..(1) and $y+2x=7$ ……(2) To find the point of intersection, substitute equation (1) in (2). $\frac{3}{x}+2x=7$ Multiplying by *x* throughout, $2x^2-7x+3=0$ $x=3$ or $x=0.5$ Hence, the *x*-coordinates of A and B are 0.5 and 3. [A2] <u>Comments</u> • Easy question which involves simple substitution of the two equations together and solve for values of *x*.	[2]

- This question does not require us to resolve for the value of y coordinates of the points of intersection.

(ii) [4]

Topic: Integration to find volume bounded by two curves

Sketching with a GC, we have:



Hence, we should take the volume generated by the line minus the volume generated by the curve between $x=\frac{1}{2}$ and $x=3$ as the line is above the curve.

Recall volume generated by rotating about *x* axis is given by $\pi\int y^2 dx$

Volume= $\pi\int_{0.5}^{3}\left(7-2x\right)^2-\left(\frac{3}{x}\right)^2 dx$ [M1]

$=\pi\int_{0.5}^{3}49-28x+4x^2-\frac{9}{x^2}dx$

$=\pi\left[49x-14x^2+\frac{4}{3}x^3+\frac{9}{x}\right]_{0.5}^{3}$ [M1]

$=\pi\left(60-39\frac{1}{6}\right)$ [M1]

$=\frac{125\pi}{6}units^3$ [A1]

<u>Comments</u>

- Careless mistakes include forgetting to multiplying by π and squaring each function separately.
- We can check our answer with the use of the GC here.
- Note that the volume generated by the line is not the volume of a cone but the volume of a frustrum. For simple evaluation of the volume generated, we can just use the integration formula.

3. (i) [3]

Topic: Differential equation involving the use of substitution

Given $y=ux^2$

Differentiating with respect to x, we have:

$\frac{dy}{dx}=x^2\frac{du}{dx}+2ux$ [M1]

From $x\frac{dy}{dx}=2y-6$, we substitute away the value of y and $\frac{dy}{dx}$.

		$x\left(x^2\dfrac{du}{dx}+2ux\right)=2ux^2-6$ $x^3\dfrac{du}{dx}+2ux^2=2ux^2-6$ $\dfrac{du}{dx}=-\dfrac{6}{x^3}$ [M1] $\therefore f(x)=-\dfrac{6}{x^3}$ [A1] Comments • To transform to the form of $\dfrac{du}{dx}=f(x)$, we have to make sure the variable y does not appear in the final equation. • We need to remember to apply product rule when differentiating $y=ux^2$ as u is also a function of x.	
	(ii)	Topic: Solving differential equation with the use of substitution From (i), $x\dfrac{dy}{dx}=2y-6$ can be rewritten as $\dfrac{du}{dx}=-\dfrac{6}{x^3}$ where $y=ux^2$. Integrating with respect to x, we have: $\int 1\,du=-\int\dfrac{6}{x^3}\,dx$ $u=\dfrac{3}{x^2}+C$ [M1] To find the value of C, we need to first find the corresponding value of u when $x=1$, $y=2$. Subst $x=1$, $y=2$ in $y=ux^2$, we have u=2 $\therefore 2=\dfrac{3}{1^2}+C\Rightarrow C=-1$ [M1] Thus, $u=\dfrac{3}{x^2}-1\Rightarrow\dfrac{y}{x^2}=\dfrac{3}{x^2}-1$ [M1] Multiplying by x^2 throughout, $y=3-x^2$ [M1]	[4]

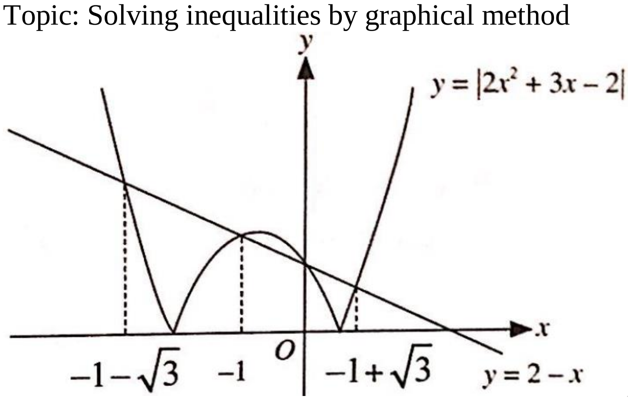
4.	(i)	Topic: Solving equations involving absolute sign Given $\left\|2x^2+3x-2\right\|=2-x$, $\therefore 2x^2+3x-2=2-x$ $2x^2+4x-4=0$ $x^2+2x-2=0$ $x=\dfrac{-2\pm\sqrt{2^2-4(1)(-2)}}{2(1)}$ [M1] $=\dfrac{-2\pm2\sqrt{3}}{2}$ $=-1-\sqrt{3}$ or $-1+\sqrt{3}$ [A1] OR $2x^2+3x-2=x-2$ $2x^2+2x=0$ $x(x+1)=0$ [M1] $x=0$ or $x=-1$ [A1]	[4]

Hence, the roots are $x = -1-\sqrt{3},\ -1+\sqrt{3},\ 0$ or -1

Comments

- For more rigorous working, we should check that the roots are valid.
- $2x^2+3x-2 = 2-x$ is valid when $2x^2+3x-2 \geq 0 \Rightarrow x \leq -2$ or $x \geq 0.5$
  Since $-1-\sqrt{3} = -2.73 \leq -2$ and $-1+\sqrt{3} = 0.732 \geq 0.5$, both roots are valid
- $2x^2+3x-2 = x-2$ is valid when $2x^2+3x-2 \leq 0 \Rightarrow -2 \leq x \leq 0.5$
  Since $x=0$ and $x=-1$ lies within the region of $-2 \leq x \leq 0.5$, these 2 roots are also valid.

(ii) Topic: Solving inequalities by graphical method [4]



[M1 for shape of curves, M1 for labelling the x coordinates of the intersection points]

Hence, for $\left|2x^2+3x-2\right| < 2-x$, we have $-1-\sqrt{3} < x < -1$ or $0 < x < -1+\sqrt{3}$ [A2]

Comments

- The question wants the values of x which satisfy the condition of the line being higher than the curve.
- The equal sign should not be included for both sets of the solution of the inequality.

5. Topic: Functions on self-inverse functions [5]

Since g(x) = x and ff = g, f must be self-inverse i.e. f = f^{-1} because we have learnt that $ff^{-1}(x) = x$. [M1]

$$f(x) = \frac{x+a}{x+b} = 1 + \frac{a-b}{x+b}$$ [M1]

Since f(x) has a horizontal asymptote of y=1, and vertical asymptote of x= -b, for it to be able to self-inverse, i.e. reflection about the $y = x$ line should get back the same function, x=1 must be an asymptote for f(x) as well.

Hence, b = -1. [M1]

Thus, $f^{-1}(x) = 1 + \frac{a+1}{x-1}, x \in \mathbb{R}, x \neq 1, a \neq -1$ (same function)

[M1 for the expression, M1 for stating the domain]

Comments

- For finding the inverse function of f, the domain should be stated for the function to be well-defined.

	• To understand why the horizontal asymptote and vertical asymptote must have the same value, we suppose they are not of the same value e.g. y=1 and x=2. The asymptote of f^{-1} then becomes y=2 and x=1 after reflection about the y=x axis. This means that f is no longer able to self-inverse. • We can also express the inverse function of f as a single fraction, $f^{-1}(x)=\frac{x+a}{x-1}$		

6	(i)	Topic: Basic properties of vectors Starting from $\mathbf{a}\times 3\mathbf{b}=2\mathbf{a}\times\mathbf{c}$ $\mathbf{a}\times 3\mathbf{b}-2\mathbf{a}\times\mathbf{c}=0$ $\mathbf{a}\times 3\mathbf{b}+\mathbf{a}\times(-2\mathbf{c})=0$ $\mathbf{a}\times(3\mathbf{b}-2\mathbf{c})=0$ [M1] Hence, $\mathbf{a}\,//\,(3\mathbf{b}-2\mathbf{c})$ [M1] Thus $3\mathbf{b}-2\mathbf{c}=\lambda\mathbf{a}$ where $\lambda\in\mathbb{R}$. Comments We have to apply the reverse of the distributive law involving vector product.	[2]
	(ii)	Topic: Scalar product of vectors Applying the technique of squaring both sides for $3\mathbf{b}-2\mathbf{c}=\lambda\mathbf{a}$, we have: $(3\mathbf{b}-2\mathbf{c})^2=\lambda^2\lvert\mathbf{a}\rvert^2$ [M1] $9\lvert\mathbf{b}\rvert^2-12\mathbf{bc}+4\lvert\mathbf{c}\rvert^2=\lambda^2$ $144-12\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\cos 60^0+4=\lambda^2$ [M2] $\lambda^2=144-(12)(4)(1)(0.5)+4=124$ $\lambda=2\sqrt{31}$ or $-2\sqrt{31}$ [A2] Comments • We will have to think about how to make use of the information provided: **a** and **c** are unit vectors, modulus of **b** is 4 and angle between b and c is 60^0 to arrive at the values of λ.	[5]

7	(i)	Topic: Differentiation Given that $\frac{x^2-4y^2}{x^2+xy^2}=\frac{1}{2}$, $2x^2-8y^2=x^2+xy^2$ Differentiate with respect to *x*, $4x-16y\frac{dy}{dx}=2x+y^2+2xy\frac{dy}{dx}$ [M1] Rearranging the equation,	[3]

$$4x-2x-y^2=2xy\frac{dy}{dx}+16y\frac{dy}{dx}$$ [M1]

$$\left(2xy+16y\right)\frac{dy}{dx}=2x-y^2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}(shown)$$ [M1]

Comments

- It is easier to cross multiply first rather than apply quotient rule directly where students may be more prone to careless mistakes.
- Keep in mind to apply chain rule and product rule along the way.

(ii) [6]

- Topic: Differentiation on finding the equations of tangents to curve
- To find the exact coordinates of N, we need to form the two equations of the tangents to C and equate them together.
- First, we find the corresponding y and $\frac{dy}{dx}$ values on C when x=1

Given $\frac{x^2-4y^2}{x^2+xy^2}=\frac{1}{2}$,when x=1,

$$\frac{1-4y^2}{1+y^2}=\frac{1}{2}\Rightarrow 2-8y^2=1+y^2$$

$$9y^2-1=0\Rightarrow y=-\frac{1}{3} \text{ or } y=\frac{1}{3}$$ [M1]

$$\frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}$$

When x=1, $y=-\frac{1}{3}$,

$$\frac{dy}{dx}=\frac{2-\frac{1}{9}}{-\frac{2}{3}-\frac{16}{3}}=-\frac{17}{54}$$ [M1]

When x=1, $y=\frac{1}{3}$,

$$\frac{dy}{dx}=\frac{2-\frac{1}{9}}{\frac{2}{3}+\frac{16}{3}}=\frac{17}{54}$$ [M1]

Thus the equation of the two tangent lines are:

$$y+\frac{1}{3}=-\frac{17}{54}\left(x-1\right) \text{ and } y-\frac{1}{3}=\frac{17}{54}\left(x-1\right)$$ [M1]

Equating the two tangent line equations together,

$$-\frac{17}{54}\left(x-1\right)-\frac{1}{3}=\frac{17}{54}\left(x-1\right)+\frac{1}{3}$$

Multiplying by 54 throughout,

$$-17x+17=17x-17+36$$

$$34x=-2$$

$$x=-\frac{1}{17}$$ [M1]

When $x=-\frac{1}{17}$, $y=-\frac{17}{54}\left(-\frac{1}{17}-1\right)-\frac{1}{3}=0$

		Hence, the coordinates of N is $\left(-\frac{1}{17}, 0\right)$ [A1] <u>Comments</u> Very manageable and standard manipulations to find the coordinates of N	

8	(i)	Topic: Sequence and Series Given that $u_1=5$, $u_2=15$, applying the recurrence relation: $u_{n+1} = 2u_n + An$ and substituting n=1, $u_2 = 2u_1 + A$ $15=(2)(5)+A$ $A=5$ [M1] Given that $u_2=15$, applying the recurrence relation $u_{n+1} = 2u_n + 5n$ again and substituting n=2, $u_3 = 2u_2 + (5)(2)$ $=30+10 = 40$ [M1] <u>Comments</u> • Just be careful with the manipulations to avoid careless mistakes.	[2]
	(ii)	Topic: Sequence and Series, solving system of linear equations Given $u_n = a(2^n) + bn + c$, $u_1 = 2a + b + c = 5$ $u_2 = 4a + 2b + c = 15$ $u_3 = 8a + 3b + c = 40$ Solving with GC, a=7.5, b=-5, c=-5 [M3 for forming the 3 linear equations, M1 for solving the values of a, b and c] <u>Comments</u> We should solve the above system of linear equations with GC, and not manually to save time.	[4]
	(iii)	Topic: Sequence and Series $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[7.5\left(2^r\right)-5r-5\right]$ $=\sum_{r=1}^{n}\left[7.5\left(2^r\right)-5r-5\right]$ $=7.5\sum_{r=1}^{n}2^r-5\sum_{r=1}^{n}r-5\sum_{r=1}^{n}1$ [M1] $=15\left(\frac{2^n-1}{2-1}\right)-5\frac{n(n+1)}{2}-5n$ $=15\left(2^n-1\right)-2.5n^2-7.5n$ [M1 for the use of sum of GP formula, M1 for expanding sum of r and sum of constant, A1for final answer] <u>Comments</u> This question requires us to recall and apply the following formulae for the series expansion. $\sum_{r=1}^{n} a^r = a\sum_{r=1}^{n} a^{r-1} = a\left(1+a+a^2+...+a^{n-1}\right)=\frac{a\left(a^n-1\right)}{a-1}$	[4]

		$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ and $\sum_{r=1}^{n} a = na$	

9	(i)	Topic: Differentiation of functions defined parametrically Given that $x = 2\theta - \sin 2\theta, y = 2\sin^2\theta$, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 4\sin\theta\cos\theta \times \frac{1}{2-2\cos 2\theta}$ $= \frac{4\sin\theta\cos\theta}{2-2\cos 2\theta}$ [M2 for applying chain rule to find $\frac{dy}{dx}$] Dividing by 4 from the top and bottom of the expression, we have: $= \frac{\sin\theta\cos\theta}{\frac{1-\cos 2\theta}{2}} = \frac{\sin\theta\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta(shown)$ [M2 for the manipulation to get the final expression] <u>Comments</u> Students need to be familiar with the double angle formula in order to get rid of the term in terms of 2θ.	[4]
	(ii)	Topic: Differentiation on finding equation of normal to curve defined parametrically Since $\frac{dy}{dx} = \cot\theta$, the gradient of the normal to the curve at the point $\theta = \alpha$ can be given as $-\tan\alpha$. [M1] We would want to form the equation of the normal to the curve at the point $\theta = \alpha$ Given $x = 2\theta - \sin 2\theta, y = 2\sin^2\theta$, The equation of the normal is: $y - y_1 = (-\tan\alpha)(x - x_1)$ $y - 2\sin^2\alpha = (-\tan\alpha)(x - 2\alpha + \sin 2\alpha)$ [M1] For the *x* coordinate of A where the normal to the curve meets the *x* axis, set y=0. $-2\sin^2\alpha = (-\tan\alpha)(x - 2\alpha + \sin 2\alpha)$ $x - 2\alpha + \sin 2\alpha = \frac{2\sin^2\alpha}{\tan\alpha}$ $x - 2\alpha + 2\sin\alpha\cos\alpha = 2\sin\alpha\cos\alpha$ $\therefore x = 2\alpha$ Where *k* is 2 [M2 for the manipulation to simplify the expression and obtain the value of *x*] <u>Comments</u> Students need to recall that the gradient of the normal is $-\frac{1}{m}$ where m is the gradient of the tangent.	[4]
	(iii)	Topic: Integration involving definite integrals and trigonometric manipulations Given that $x = 2\theta - \sin 2\theta, y = 2\sin^2\theta$,	[5]

$$\int_{\beta}^{\gamma}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_{\beta}^{\gamma}\sqrt{\left(2-2\cos 2\theta\right)^2+\left(4\sin\theta\cos\theta\right)^2}\,d\theta$$ [M1 for substitution]

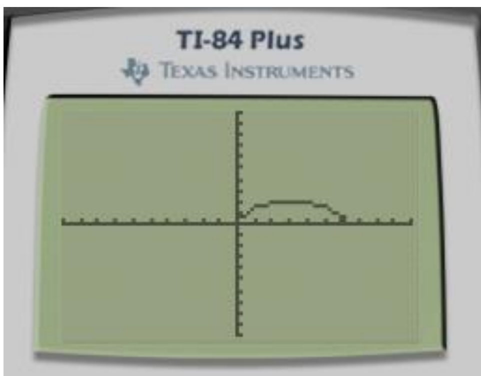
$$=\int_{\beta}^{\gamma}\sqrt{16\left(\frac{1-\cos 2\theta}{2}\right)^2+\left(4\sin\theta\cos\theta\right)^2}\,d\theta$$

$$=\int_{\beta}^{\gamma}\sqrt{16\sin^4\theta+16\sin^2\theta\cos^2\theta}\,d\theta$$

$$=4\int_{\beta}^{\gamma}\sqrt{\sin^2\theta\left(\sin^2\theta+\cos^2\theta\right)}\,d\theta$$

$$=4\int_{\beta}^{\gamma}\sin\theta\,d\theta=-4\left[\cos\theta\right]_{\beta}^{\gamma}$$

[M3 for the manipulation to simplify the expression]



From GC, this is the shape of the curve C where the left end is when θ=0 and right end is when θ=π. Hence, β=0 and γ=π

Hence, length = $-4\left[\cos\theta\right]_0^{\pi}=-4\left(-1-1\right)=8\,units$ [M1]

<u>Comments</u>

- Again, you need to be familiar with double angle formulae and apply here to simplify your expression.
- Some students may leave their answer in terms of β and γ which is insufficient for answering this question as we can continue to obtain the exact total length of C.

10. (i) [2]

Topic: Implicit Differentiation

Given $L\frac{dI}{dt}+RI+\frac{q}{C}=V$,

Differentiating with respect to t, keeping in mind R, C and L are constants,

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\frac{dq}{dt}=\frac{dV}{dt}$$

Condition: If V is a constant, $\frac{dV}{dt}=0$. [M1]

Also, since $I=\frac{dq}{dt}$, we have:

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=0(shown)$$ [M1]

(ii) [5]

Topic: Differentiation

Instead of trying to integrate the differential equation and arrive at the solution $I=Ate^{-\frac{Rt}{2L}}$ which is very tedious, it is easier to use the solution and differentiate it. Then we substitute the relevant expressions into the second order differential equation shown above.

$I=Ate^{-\frac{Rt}{2L}}$

Differentiating with respect to t, applying product rule,

$$\frac{dI}{dt}=Ae^{-\frac{Rt}{2L}}-\frac{AR}{2L}te^{-\frac{Rt}{2L}}$$ [M1]

Differentiating with respect to t again,

$$\frac{d^2I}{dt^2}=-\frac{AR}{2L}e^{-\frac{Rt}{2L}}-\frac{AR}{2L}e^{-\frac{Rt}{2L}}+\frac{AR^2}{4L^2}te^{-\frac{Rt}{2L}}$$

$$=-\frac{AR}{L}e^{-\frac{Rt}{2L}}+\frac{AR^2}{4L^2}te^{-\frac{Rt}{2L}}$$ [M1]

Substituting the expression for I, $\frac{dI}{dt}$ and $\frac{d^2I}{dt^2}$ into $L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=0$,

$$-ARe^{-\frac{Rt}{2L}}+\frac{AR^2}{4L}te^{-\frac{Rt}{2L}}+ARe^{-\frac{Rt}{2L}}-\frac{AR^2}{2L}te^{-\frac{Rt}{2L}}+\frac{Ate^{-\frac{Rt}{2L}}}{C}=0$$

Observe that the first and third term can be cancelled, second and fourth term can be combined.

$$-\frac{AR^2}{4L}te^{-\frac{Rt}{2L}}+\frac{Ate^{-\frac{Rt}{2L}}}{C}=0$$

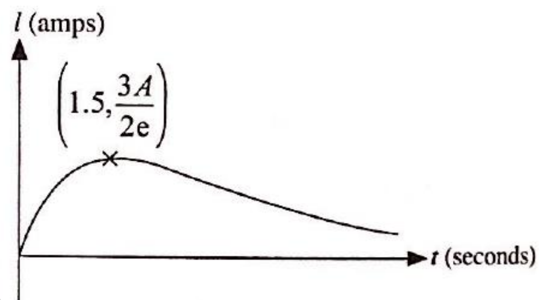
Dividing by $Ate^{-\frac{Rt}{2L}}$ throughout,

$$-\frac{R^2}{4L}+\frac{1}{C}=0$$

Hence, $C=\frac{4L}{R^2}(shown)$

[M3 for the manipulation to the final answer]

		<u>Alternative approach for the differentiation</u> $I = Ate^{-\frac{Rt}{2L}}$ Differentiating with respect to t, applying product rule, $\frac{dI}{dt} = Ae^{-\frac{Rt}{2L}} - \frac{AR}{2L}te^{-\frac{Rt}{2L}} = \frac{I}{t} - \frac{R}{2L}I$ Differentiating with respect to t again, $\frac{d^2I}{dt^2} = -\frac{I}{t^2} + \frac{1}{t}\frac{dI}{dt} - \frac{R}{2L}\frac{dI}{dt} = -\frac{I}{t^2} + \left(\frac{1}{t} - \frac{R}{2L}\right)\frac{dI}{dt}$ $= -\frac{I}{t^2} + I\left(\frac{1}{t} - \frac{R}{2L}\right)^2$ Substituting the values into $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$ $L\left[-\frac{I}{t^2} + I\left(\frac{1}{t} - \frac{R}{2L}\right)^2\right] + R\left(\frac{I}{t} - \frac{R}{2L}I\right) + \frac{I}{C} = 0$ $L\left[-\frac{I}{t^2} + \frac{I}{t^2} - \frac{RI}{tL} + \frac{R^2I}{4L^2}\right] + R\left(\frac{I}{t} - \frac{R}{2L}I\right) + \frac{I}{C} = 0$ $-\frac{RI}{t} + \frac{R^2I}{4L} + \frac{RI}{t} - \frac{R^2I}{2L} + \frac{I}{C} = 0$ $-\frac{R^2I}{4L} + \frac{I}{C} = 0$ $\therefore C = \frac{4L}{R^2}(shown)$ <u>Comments</u> • For some students, they may try to integrate the differential equation and get stuck halfway through the manipulations. • Even if you had used the easier method of differentiation, it is still mathematically tedious and intimidating with the sheer amount of arbitrary values. We just have to persevere through the manipulations.	
	(iii)	Topic: Differentiation which involves the solving of a maxima problem Previously, we have found $\frac{dI}{dt} = Ae^{-\frac{Rt}{2L}} - \frac{AR}{2L}te^{-\frac{Rt}{2L}}$ and $\frac{d^2I}{dt^2} = -\frac{AR}{L}e^{-\frac{Rt}{2L}} + \frac{AR^2}{4L^2}te^{-\frac{Rt}{2L}}$. Substituting the values of R=4, L=3 and C=0.75 given in this part of the question into the equations, $\frac{dI}{dt} = Ae^{-\frac{2t}{3}} - \frac{2}{3}Ate^{-\frac{2t}{3}} = Ae^{-\frac{2t}{3}}\left(1 - \frac{2}{3}t\right)$ [M1] For $\frac{dI}{dt} = 0, t = 1.5$ [M1]	[4]

		Subst t=1.5 in the equation of I where $I = Ate^{-\frac{Rt}{2L}}$, $I = 1.5Ae^{-1} = \frac{3A}{2e}$ [M1] $\frac{d^2I}{dt^2} = -\frac{4A}{3}e^{-\frac{2t}{3}} + \frac{16A}{(4)(9)}te^{-\frac{2t}{3}} = -\frac{4A}{3}e^{-\frac{2t}{3}} + \frac{4A}{9}te^{-\frac{2t}{3}}$ When t=1.5, $\frac{d^2I}{dt^2} = -\frac{4A}{3e} + \frac{4A}{9e}\left(\frac{3}{2}\right) = -\frac{2A}{3e} < 0$ [M1] Hence, this verifies that maximum value of I is $\frac{3A}{2e}$ when t=1.5. <u>Comments</u> • Since the question has requested us to find the maximum value of I in terms of A, we should not be trying to substitute the values of R, L and C into the differential equations of $L\frac{dI}{dt} + RI + \frac{q}{C} = V$ and $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$ provided in the beginning of the question as they would not allow us to obtain the required value.	
	(iv)	Sketch the graph of *I* against *t*. Topic: Graph Sketching  [M1 for shape, M1 for labelling the axis and stationary point] <u>Comments</u> • We use the results obtained in the earlier parts of the question to sketch the graph. • The graph has to pass through the origin as the question stated that I=0 when t=0. • In (iii), we have found that the maximum value of I is $\frac{3A}{2e}$ when t=1.5.	[2]

11	(i) (a)	Topic: Geometric Series Value at the end of 31 December 2016 $=\$100(1.002)^{12}=\$102.43$ <u>Comments</u> • We need to recognise that for a savings plan which yield compound interest, it is an application of the geometric progression formula. • Be careful that a is 0.2% so the common ratio should be 1.002. Getting this right is very crucial as it will affect the answer we obtain for the following parts of the question.	[1]
	(b)	Topic: Summation of Geometric Series Total amount in the account at the end of 31 December 2016 $=100(1.002)^{12}+100(1.002)^{11}+....+100(1.002)^{1}$ [M1] $=100(1.002)\left[1+1.002+...+1.002^{11}\right]$ $=(100.2)\left(\frac{1.002^{12}-1}{1.002-1}\right)$ [M1] $=\$1215.71$ [A1] <u>Comments</u> • To obtain the expression for the total amount at the end of 31 December 2016, we need to know that the amount invested in January will earn interest 12 times, the amount invested in February will earn interest 11 times and so on. • After that, we have to apply the formula for the summation of geometric series correctly, the value of n is 12 here as there are 12 terms.	[3]
	(c)	Topic: Summation of geometric series Value at the end of nth month $(100.2)\left(\frac{1.002^{n}-1}{1.002-1}\right)>3000$ [M1] $1.002^{n}-1>\frac{10}{167}\Rightarrow n\lg 1.002>\lg\frac{177}{167}$ $n>29.1$ [M1] Hence, the value at the end of 30th month $=(100.2)\left(\frac{1.002^{30}-1}{1.002-1}\right)=\$3094.82$ [M1] At the beginning of the 30th month, the value is $\frac{3094.82}{1.002}=\$3088.64$ [M1] Hence, the month for the total amount to first exceed \$3000 will be the 30th month and it occurs at the beginning of the month. In other words, it first exceeds \$3000 on 1 June 2018. [M1]	[5]

		Alternative approach after finding n > 29.1 The value at the end of 29th month $=(100.2)\left(\dfrac{1.002^{29}-1}{1.002-1}\right)=\$2988.64$ At the beginning of the 30th month, the value is 2988.64 + 100 = \$3088.64. Hence, the month for the total amount to first exceed \$3000 will be the 30th month and it occurs at the beginning of the month. In other words, it first exceeds \$3000 on 1 June 2018. Comments • Since we have found the expression for the total amount at the end of 12 months, it is easier to just modify the expression to find the total amount at the end of n months, rather than start afresh and find the expression for total amount at the beginning of n months. • We should not just state the number of months, but go on to find the exact date, month and year.	
(ii)	(a)	Find, in terms of b, how much \$100 invested on 1 January 2016 will be worth at the end of 31 December 2016. Topic: Arithmetic Series Value at the end of 31 December 2016 $=100+12b$ Comments • We should recognise that for \$100 invested on 1 January 2016, Mr Wong will earn the interest of \$b 12 times by the end of 31 December 2016.	[1]
	(b)	Topic: Summation of Arithmetic Series Value at the end of 31 December 2017 $=(100)(24)+24b+23b+...+b$ [M1] $=2400+\dfrac{24}{2}[2b+(23)(b)]$ [M1] $=2400+300b$ For the total value of all investments including bonuses to be worth \$2800 at the end of 31 December 2017, $2400+300b=2800 \Rightarrow b=\dfrac{4}{3}=\$1.33$ Comments • To form the expression for the value at the end of 31 December 2017, we need to recognise that Mr Wong would have invested \$100 24 times in 2 years. The amount of \$100 invested on 1 January 2016 would earn interest of \$b 24 times. The amount of \$100 invested on 1 February 2016 would earn interest of \$b 23 times and so on. • We should leave our final answer in 2 decimal places instead of exact form since the question involves money.	[3]

	(iii)	Topic: Arithmetic and Geometric Series The value for plan P at the end of the 60th month $=(101)\left(\frac{1.01^{60}-1}{1.01-1}\right)=\$8248.63665$ [M1] The value for plan Q at the end of 60th month $=6000+\frac{60}{2}\left[2b+(59)(b)\right]$ [M1] For the same total value, $6000+\frac{60}{2}\left[61b\right]=8248.63665$ $b=1.2287=\$1.23(2d.p.)$ [A1] Comments • The expression for the value of plan P is similar to $(100.2)\left(\frac{1.002^{n}-1}{1.002-1}\right)$, just that we substitute a=1 instead of a=0.2, substitute n as 60. • The expression for the value of plan Q is similar to $2400+\frac{24}{2}\left[2b+(23)(b)\right]$, just that there is 60months now so Mr Wong would have invested \$6000 instead of \$2400. He would also have earned greater amount of interest.	[3]

End of Solutions